Marmara University
Faculty of Engineering
Math 101.1/Math 1001.1
Final Exam

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Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	10	10	15	15	10	10	10	10	100
Score:										

NAME: ______________________ STUDENT NO: ______________________

Give detailed work.

SIGNATURE: ______________________

1. (10 points) Evaluate $\int \frac{8dx}{x^3+4x}$.

$$\frac{8}{x^3+4x} = \frac{8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$8 = x^2(A+B) + xC + 4A \Rightarrow A=2,\ C=0,\ B=-2.$$

$$\int \frac{2}{x}\,dx = 2\ln|x| + C = \ln x^2 + C$$

$$\int \frac{-2x}{x^2+4}\,dx \underset{\substack{u=x^2+4 \\ du=2xdx}}{=} \int -\frac{du}{u} = -\ln|u| + C = -\ln|x^2+4| + C$$

$$\int \frac{8\,dx}{x^3+4x} = \int \frac{2}{x}\,dx + \int \frac{-2x\,dx}{x^2+4} = \ln x^2 - \ln(x^2+4) + C = \ln\left(\frac{x^2}{x^2+4}\right) + C$$

2. (10 points) If

$$f(x) = -\ln x + \int_0^x e^{\arctan t}\,dt$$

find $f'(1)$

$$f'(x) = -\frac{1}{x} + e^{\arctan x}$$

$$f'(1) = -1 + e^{\arctan 1} = -1 + e^{\pi/4}.$$

6. (10 points) Find

$$\lim_{x\to 0+}(1+\tan x)^{\frac{1}{x}}$$

$$y=(1+\tan x)^{1/x} \Rightarrow \ln y = \frac{\ln(1+\tan x)}{x}$$

$$\lim_{x\to 0+} \ln y = \left[\frac{0}{0}\right] \quad \text{Use L'hopital's rule.}$$

$$\lim_{x\to 0+} \ln y = \lim_{x\to 0+} \frac{\frac{\sec^2 x}{1+\tan x}}{1} = \lim_{x\to 0+} \frac{\sec^2 x}{1+\tan x} = \frac{1}{1}$$

$$\lim_{x\to 0+}(1+\tan x)^{1/x} = e^1 = e.$$

7. (10 points) Is

$$f(x)=\begin{cases}3x+2x^2\sin\left(\frac{1}{x}\right), & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

differentiable at $x=0$. If so, find $f'(0)$.

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{3h+2h^2\sin\left(\frac{1}{h}\right)}{h}$$

$$= \lim_{h\to 0} 3+2h\sin\left(\frac{1}{h}\right)$$

$$-1 \leq \sin\left(\frac{1}{h}\right) \leq 1 \Rightarrow -|h| \leq h\sin\left(\frac{1}{h}\right) \leq |h|$$

Since $\lim_{h\to 0} -|h| = \lim_{h\to 0} |h| = 0$, by Sandwich Theorem

$$\lim_{h\to 0} h\sin\left(\frac{1}{h}\right)=0$$

So $f'(0)=3$.

5. (15 points) Let $f(x) = x^4 - 4x^2 + 5$ for $-2 \le x \le 2$. Sketch the graph of $f(x)$ by making a table which shows

(a) the intervals on which the function is increasing or decreasing and the local extremum values, and

(b) the intervals on which the function is concave up or down and the inflection points.

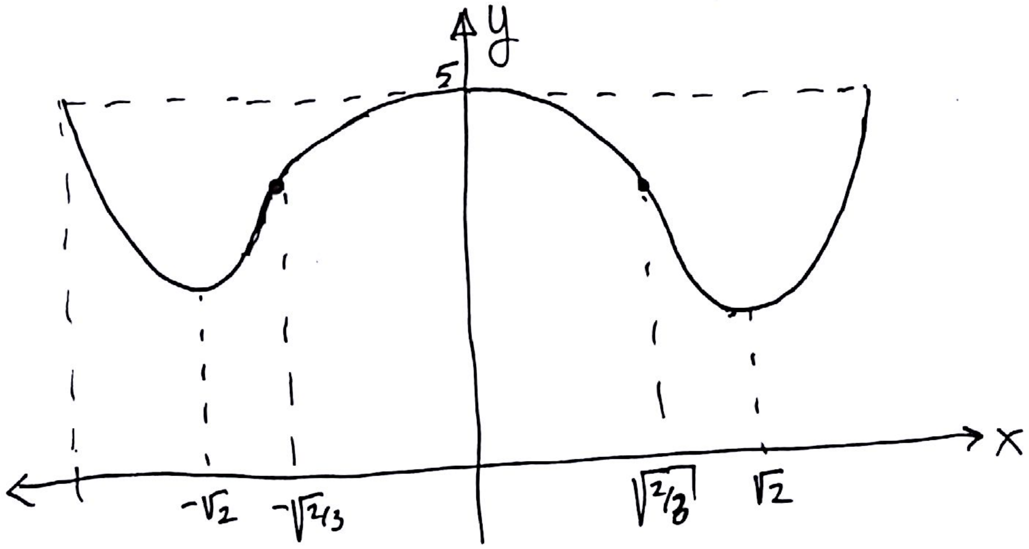
$$f'(x) = 4x^3 - 8x = 4x(x^2 - 2)$$

$$f'(x) = 0 \Leftrightarrow x = 0, \quad x = \mp\sqrt{2}.$$

$$f''(x) = 12x^2 - 8 = 4(3x^2 - 2)$$

$$f''(x) = 0 \Leftrightarrow x = \mp\sqrt{2/3}.$$

x	-2		$-\sqrt{2}$		$-\sqrt{2/3}$		0		$\sqrt{2/3}$		$\sqrt{2}$		2
$f'(x)$		$-$	0	$+$		$+$	0	$-$		$-$	0	$+$	0
$f''(x)$		$+$		$+$	0	$-$		$-$	0	$+$		$+$	
$f(x)$		$\searrow \cup$		$\nearrow \cup$		$\nearrow \cap$		$\searrow \cap$		$\searrow \cup$		$\nearrow \cup$	
	max		min		infl.		max		infl.		min		max



$$f(\mp 2) = 16 - 16 + 5 = 5$$

$$f(\mp\sqrt{2}) = 4 - 8 + 5 = 1$$

$$f(\mp\sqrt{2/3}) = \frac{4}{9} - 4 \cdot \frac{2}{3} + 5 = \frac{4 - 24 + 45}{9} = 25/9.$$

3. (10 points) Evaluate $\int x\sec^2 x\,dx$.

$u = x$, $dv = \sec^2 x\,dx \rightarrow du = 1\,dx$, $v = \tan x$

$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx$

$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int -\frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C.$

$u = \cos x$
$du = -\sin x\,dx$

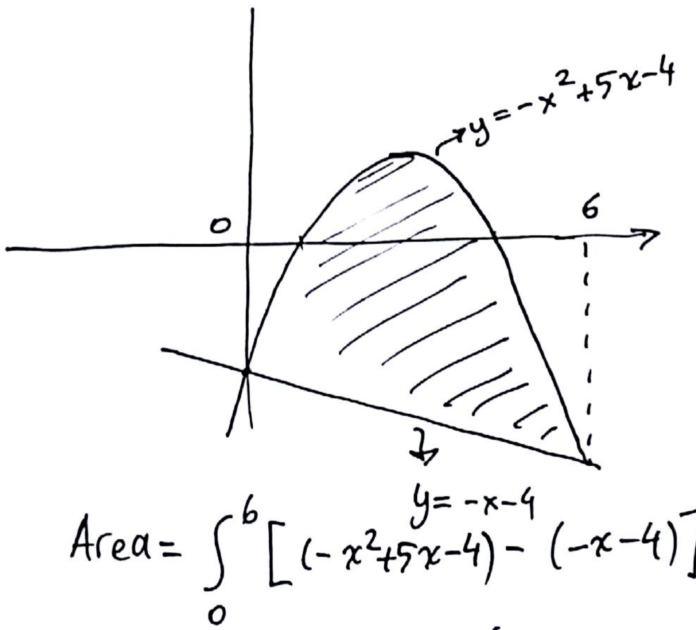
$\int x\sec^2 x\,dx = x\tan x + \ln|\cos x| + C$

4. (15 points) Find the area of the bounded region lying between the curves $y = -x^2+5x-4$ and $y = -x-4$.

$y = -x^2+5x-4 = -(x^2-5x+4) = -(x-4)(x-1)$

Two curves intersect at

$-x^2+5x-4 = -x-4 \Rightarrow x^2-6x=0 \Rightarrow x=0,\ x=6.$



$$\text{Area} = \int_0^6 [(-x^2+5x-4) - (-x-4)]\,dx = \int_0^6 (-x^2+6x)\,dx$$

$$= -\frac{x^3}{3} + 3x^2\Big|_0^6 = -\frac{6^3}{3} + 3\cdot 6^2 = 6^3\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{6^3}{6} = 36$$

8. (10 points) Find the equation of tangent line to the curve

$$3^y - \log_3 x = y^2 + 1$$

at (1,0).

Take $\frac{d}{dx}$ of both sides. Let $y' = \frac{dy}{dx}$.

$$3^y \ln 3 \cdot y' - \frac{1}{x \ln 3} = 2yy'$$

At $x=1, y=0$,

$$\ln 3 \, y' - \frac{1}{\ln 3} = 0 \Rightarrow y' = \frac{1}{(\ln 3)^2} = \text{slope of the tangent line.}$$

Equation: $y - 0 = \frac{1}{(\ln 3)^2}(x-1) \Rightarrow \boxed{y = \frac{x-1}{(\ln 3)^2}}$

9. (10 points) A hotel finds that it can rent 200 rooms per day if it charges 40TL per day. For each 1TL increase in rental rate 4 fewer rooms will be rented per day. To maximize revenues, how much should be charged for each room per day?

Let (room rent) $= 40 + x$.

Then (number of rooms rented) $= 200 - 4x$

Revenue $= R = (40+x)(200-4x)$.

$R'(x) = (200-4x) - 4(40+x) = 40 - 8x$

$R'(x) = 0 \Leftrightarrow x = 5$.

x	0		5	
$R'(x)$		+		−
$R(x)$		↗	max	↘

Revenue is maximized at $x=5$.

So room rent should be 45TL.